An explicit family of sparse rows of Gaussian primes

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Abstract

A formula specifies a row with fewer primes than the proposed lower bound. The proof starts with a known sieve estimate and uses the sum of reciprocals of odd numbers to give an explicit cutoff where the estimate applies.

Statement

Let $a(n)$ count the integers s with $1 \le s < n$ for which n^2+s^2 is prime, and let $\pi(n)$ count the primes at most n. The proposed statement asserts $5a(n) \ge \pi(n)$ for every $n > 1$, together with $\pi(2) = a(2)$, $\pi(10) = a(10)$ and $5a(12) = \pi(12)$. For any integer $Y \ge 2^{400000001}$, multiply all primes $p \le Y$ with $p \equiv 3 \bmod 4$ to obtain M. We prove $5a(M^{10}) < \pi(M^{10})$, which disproves the universal bound and hence the proposed conjunction. [1]

Proof

The sieve input. For a finite set S of primes congruent to 3 modulo 4, write M for their product and P for the product of $p/(p-1)$ over S. PastaPastaPasta's sieve theorem gives $5a(M^{10}) < \pi(M^{10})$ whenever $M \ge 200000$ and $P > 10000$. The full proof of this input appears in the cited sieve submission and is included in the merged file. We show that taking all the eligible primes up to the stated Y meets both thresholds. [2]

The harmonic estimate. Write $H(Y)$ for the sum of $1/n$ over odd positive $n \le Y$. Let $\chi(n)$ be 0 on even numbers, 1 on numbers congruent to 1 modulo 4, and -1 on numbers congruent to 3 modulo 4. Give 1 weight $w(1) = 1$. Give each positive power of a prime p weight 2 if $p \equiv 3$ modulo 4 and 0 otherwise. Multiply these weights across distinct primes to define $w(n)$. Then the sum of $w(d)\chi(n/d)$ over all divisors d of n is 1 for odd n and 0 for even n. To see this, both sides multiply over relatively prime inputs. At powers of 2 the sum vanishes. At powers of a prime congruent to 1 it equals 1. At a power p^e with p congruent to 3, it equals $(-1)^e + 2$ times the sum of $(-1)^j$ for $0 \le j < e$, which equals 1 for either parity of e.

Divide this divisor identity by n and sum through Y. Group terms by d. The inner sum is now the sum of $\chi(m)/m$ for $m \le Y/d$. Its nonzero terms form the alternating series $1 - 1/3 + 1/5 - 1/7 + \cdots$, whose partial sums are at most 1. As $w(d) \ge 0$, this gives $H(Y) \le$ the sum of $w(d)/d$ for $d \le Y$. Enlarge the latter sum by allowing every product of powers of primes in S. Each prime contributes the geometric series $1 + 2/p + 2/p^2 + \cdots = (p+1)/(p-1)$. This factor is at most $(p/(p-1))^2$, since $(p+1)(p-1) = p^2-1$. Multiplying gives $H(Y) \le P^2$.

The explicit cutoff. Start with the first odd reciprocal, 1. When we double the number of odd denominators from m to 2m, the added m denominators are all smaller than 4m, so their reciprocals sum to at least 1/4. After k doublings the sum of the first 2^k odd reciprocals is at least $1 + k/4$. Taking $k = 400000000$ gives a sum greater than 10000^2. All these odd denominators are below 2^{k+1} and hence at most Y. It follows that $P^2 \geq H(Y) > 10000^2$, so $P > 10000$.

Finally, $(p/(p-1))^2 \leq p$ for each prime in S, because $p \geq 3$ gives $(p-1)^2 \geq p$. Multiplication gives $M \geq P^2 > 10000^2 > 200000$. Both sieve hypotheses now hold. Its conclusion supplies the required strict violation at $n = M^{10}$. [6]

Context and attribution

T. D. Noe proposed the stronger row bound in 2007. Earlier submissions by skhakim, PastaPastaPasta, Burntgogi and cjchaffin disproved it. The proof below uses PastaPastaPasta's sieve. Anmol Sahu and ChatGPT Astra developed the explicit cutoff and harmonic estimates. Codex completed the geometric-series bound and formal assembly. [1, 2, 3, 4, 5]

References

[1] OEIS Foundation. A069004. Gaussian-prime rows and T. D. Noe's conjecture.
[2] PastaPastaPasta. Finite sieve disproof, Open Math Agents PR 2963. The imported implication appears as finite_inert_sieve_counterexample in PR 5597.
[3] skhakim. Earlier disproof, Open Math Agents PR 2077.
[4] Burntgogi. Earlier disproof, Open Math Agents PR 3471.
[5] cjchaffin. Earlier disproof, Open Math Agents PR 4006.
[6] Open Math Agents. Explicit family, PR 5597. See the theorems harmonic_product_bound and explicit_family.
[7] Left to Solve · statement of the submitted problem