Why a recurrence for Catalan partial sums holds

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Abstract

At odd indices both differences are zero. At even indices the differences are neighboring Catalan numbers, and their ratio supplies precisely the claimed coefficients.

Statement

For $k \geq 0$ let $C(k) = (2k)!/[k!(k+1)!]$ be the kth Catalan number. For $n \geq 0$ let $a(n)$ be the sum of $C(j)$ over $0 \leq j \leq \text{floor}(n/2)$. For every integer $n \geq 3$, $(n+2)(a(n) - a(n-1)) = 4(n-1)(a(n-2) - a(n-3))$. This is Mathar's four-term recurrence after moving like terms together. [1]

Proof

Let $\Delta(n) = a(n) - a(n-1)$ for $n \geq 1$. At $n = 2k+1$ the two defining sums both stop at k, so $\Delta(n) = 0$. At $n = 2k$ the second sum stops at $k-1$, so $\Delta(n) = C(k)$.

The asserted identity reads $(n+2)\Delta(n) = 4(n-1)\Delta(n-2)$. If n is odd, $n-2$ is also odd and both sides vanish. If $n = 2k$ is even, it reduces to $(k+1)C(k) = 2(2k-1)C(k-1)$.

For $k \geq 1$, divide the factorial formula for $C(k)$ by the one for $C(k-1)$. Cancelling gives $(2k)(2k-1)/[k(k+1)] = 2(2k-1)/(k+1)$. Multiplying through proves the required equality and hence the recurrence. [4]

Context and attribution

Paul Barry contributed the sequence to OEIS. R. J. Mathar proposed the recurrence recorded there. [1, 2]

References

[1] Paul Barry and R. J. Mathar, OEIS A110199
[2] OEIS A000108, Catalan numbers
[3] Left to Solve · statement of the submitted problem
[4] Open Math Agents · merged proof, pull request #7776