An infinite family for Sándor's totient–psi inequality

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Abstract

László Tóth's example at 39270 extends to infinitely many counterexamples. Doubling this even input preserves its prime divisors and doubles both sides of the computation.

Statement

Write $\varphi(n)$ for the number of integers from 1 through n coprime to n, and $\psi(n) = n$ times the product of $(1 + 1/p)$ over all distinct prime divisors p of n. Sándor asked whether $\varphi(\psi(n)) \le n$ for every integer $n \ge 2$. We prove that $\varphi(\psi(39270 \cdot 2^k)) = 41472 \cdot 2^k$ for every integer $k \ge 0$. [1]

Proof

Set $n_0 = 39270 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17$. Because n_0 is squarefree, multiplying it by the factors $(1+1/p)$ in ψ replaces each p by p+1. Hence $\psi(n_0) = 3 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdot 18 = 124416$. For $k \ge 0$ the integer $n_0 \cdot 2^k$ has exactly the same distinct prime divisors as n_0. Its psi value therefore acquires precisely the extra factor 2^k.

Now $124416 = 2^9 \cdot 3^5$. Of the integers up to $2^{9+k} \cdot 3^5$, exactly the proportion $(1-1/2)(1-1/3) = 1/3$ are coprime to it. This follows, for example, by counting the two coprime residue classes 1 and 5 in each block of six consecutive integers. Thus $\varphi(\psi(n_0 \cdot 2^k)) = (124416/3) \cdot 2^k = 41472 \cdot 2^k$.

Since 41472 exceeds 39270 by 2202, the resulting count exceeds its original input by $2202 \cdot 2^k$ for every $k \ge 0$. The family already refutes the universal inequality at $k = 0$. [3]

Context and attribution

József Sándor posed the inequality in 1988. In Remark 3.26 of his 2005 paper he credits László Tóth for 39270 and lists its first double, 78540, among counterexamples found by Kovács Lehel István. Michel Marcus contributed the OEIS record. The merged Lean proof gives the exact value at every doubling. [1, 2]

References

[1] OEIS Foundation. A196200. Sequence definition, conjecture and contributor history.

[2] József Sándor. On the Composition of Some Arithmetic Functions, II. JIPAM 6 (2005), Article 73. See equation (1.8) and Remark 3.26 for the question and Tóth's example.

[3] Open Math Agents. Merged proof submission 7807 and verification history.

[4] Left to Solve · statement of the submitted problem