Infinitely many exceptions to a rule for shifted powers

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Abstract

At n = 3438721 the two powers are both 3 modulo 487891. Their remainders recur every 5366790 steps, and that step preserves the progression in the conjecture.

Statement

For n ≥ 1 let a(n) be the smallest positive integer m such that $2^n - m$ and $3^n - m$ have greatest common divisor 1. Clark Kimberling proposed that a(n) = 3 exactly when n = 330k − 209 for some integer k ≥ 1. We disprove the implication from this arithmetic progression to a(n) = 3. [1]

Proof

Put d = 487891, u = 3438721 and h = 5366790. The required four remainders modulo d are $2^u \equiv 3$, $3^u \equiv 3$, $2^h \equiv 1$ and $3^h \equiv 1$. These are finite calculations: start with a running product of 1, read the exponent from its least significant binary digit, and multiply the product by the current base whenever that digit is 1. At each digit square the base and reduce both values modulo d. For example, the first congruence can be checked directly by evaluating pow(2, 3438721, 487891) in Python. The merged proof verifies all four remainders in Lean.

For every integer t ≥ 0, put n = u + ht. For either base b = 2 or b = 3, the identity $b^n = b^u (b^h)^t$ gives $b^n \equiv 3$ modulo d. Thus d divides both $2^n - 3$ and $3^n - 3$. Since d > 1, their greatest common divisor cannot be 1. The choice m = 3 fails the defining test.

Finally, u = 330 · 10421 − 209 and h = 330 · 16263. Substitution yields n = 330(10421 + 16263t) − 209, so each of these exponents satisfies the proposed progression condition. The first one disproves the implication, and the positive step h produces infinitely many distinct exceptions. [4]

Context and attribution

Clark Kimberling proposed the rule in OEIS A214074 in 2012. The counterexample was found by searching for compatible multiplicative orders. The contributor to PR 6206 independently checked the first witness and acknowledged the earlier public proof on this account. [1, 2, 3]

References

[1] OEIS Foundation. A214074. Sequence definition, conjecture and contributor history.
[2] Open Math Agents. Independent check and provenance discussion, PR 6206.
[3] Open Math Agents. Formal statement contributed by niclorusso, PR 5598.
[4] Open Math Agents. Merged proof submission 6211 and verification history.
[5] Left to Solve · statement of the submitted problem