The arithmetic derivative of a hyperfactorial

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Abstract

Dividing the arithmetic derivative of a positive number by that number changes the product rule into an addition rule. The exponent k in each factor k raised to k then cancels a denominator k.

Statement

Let $H(0) = 1$ and let $H(n)$ be the product of k raised to k over $1 \le k \le n$. For $m \ge 1$ the arithmetic derivative $D(m)$ is the sum of $e \times (m/p)$ over the distinct prime divisors p of m, where e is the exponent of p in m. The empty sum gives $D(1) = 0$. For every $n \ge 0$, $D(H(n)) = H(n)$ times the sum of $D(k)$ over $1 \le k \le n$. [1]

Proof

Write $L(m) = D(m)/m$. By the definition of D, this is the sum of e/p over the primes p in m. In a product ab, the exponent of each prime is the sum of its exponents in a and b. Hence $L(ab) = L(a) + L(b)$. This also applies if one factor is 1, because $L(1) = 0$.

For $k \ge 1$, the product of k copies of k has L-value $k\,L(k) = D(k)$. Apply the addition rule to all the factors of $H(n)$. The resulting identity is $L(H(n))$ equal to the sum of $D(k)$ for $1 \le k \le n$. Multiplication by $H(n)$ gives the theorem. The empty product at $n = 0$ gives zero on both sides.

For example, $H(3) = 2^2 \times 3^3 = 108$. The derivative formula gives $D(108) = 2 \times 54 + 3 \times 36 = 216$. The right side gives $108 \times (0 + 1 + 1) = 216$ as well. [5]

Context and attribution

Matthew Campbell contributed the hyperfactorial derivative sequence to OEIS. The identity in its formula section relates it to the hyperfactorials and the partial sums of the arithmetic derivative. [1, 2, 3]

References

[1] Matthew Campbell, OEIS A260619, hyperfactorial derivative
[2] OEIS A002109, hyperfactorials
[3] OEIS A003415, arithmetic derivative
[4] Left to Solve · statement of the submitted problem
[5] Open Math Agents · merged proof, pull request #9122