Reading each row of Rule 157 as a binary number

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Abstract

The left edge of the changing pattern moves one step at a time. Away from that edge the bits already have their eventual colors. This gives a simple update for the binary number and proves both of Barker's formulas.

Statement

Begin with a single on cell at position 0 of the integer line. At time n read cells $-n$ through n as a binary integer a(n), with cell $-n$ the most significant bit. For $n \geq 5$, $a(n)=3a(n-1)+14a(n-2)-48a(n-3)+32a(n-4)$. Its ordinary generating function is $(1-16x^2+32x^3-32x^4)/[(1-x)(1-2x)(1-4x)(1+4x)]$. The evolution is Wolfram Rule 157. [1]

Proof

Rule 157 gives 1 on triples 000, 010, 011, 100, 111 and 0 otherwise. At every positive time n, every cell with position $i<-n$ or $i\geq 0$ is on. Cell $-n$ is off. Between $-n$ and 0 a cell is on precisely at even positions. The first update gives this description with $n=1$.

To verify the induction, far to the left 111 remains 1. The new left edge at $-n-1$ reads 110 and turns off. The old edge at $-n$ reads 100 when n is even and 101 when n is odd, giving its required parity. At $-n+1$ an even center reads 010 or 011 and stays on; an odd center reads 001 and stays off. The remaining strictly negative interior sites alternate, with 010 staying on and 101 staying off. At 0 the last two entries are 11, so the result is on; positive sites also stay on.

The new rightmost bit is 1, and moving the right endpoint one place doubles every old positional weight. The one possible correction is the old left endpoint: when n is even it turns on with weight 2^{2n+1}. Call this correction E(n), taking $E(n)=0$ for odd n. For every $n \geq 1$ we have $a(n+1)=2a(n)+1+E(n)$.

Since $E(n+2)=16E(n)$, subtracting sixteen times the update at n from the update at $n+2$ gives $a(n+3)-2a(n+2)-16a(n+1)+32a(n)=-15$. Subtract the same equality at the preceding index to eliminate -15. Rearranging gives the claimed recurrence, starting at index five.

Let A(x) have coefficient a(n) on x^n. Multiplication by the proposed denominator combines its coefficients according to that recurrence. All coefficients above degree four vanish. The first five row values 1, 3, 7, 47, 95 yield the remaining polynomial $1-16x^2+32x^3-32x^4$. This proves the generating function. [5]

Context and attribution

Robert Price contributed OEIS A263806. Colin Barker proposed the recurrence and generating function in 2016 and 2019. The Lean statement is due to snoozingnewt; an earlier project submission also addressed this target. [1, 2, 3]

References

[1] R. Price and C. Barker, OEIS A263806
[2] snoozingnewt, Lean statement of A263806, submission 959
[3] Earlier project submission on A263806, 6402
[4] Left to Solve · statement of the submitted problem
[5] Open Math Agents · merged proof, pull request #6450