Why the middle column of Rule 158 repeats 1110

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Abstract

The central cell is on three times and off once, in a cycle that starts at time zero. This determines both the recurrence for the decimal numbers and their generating function.

Statement

Begin Rule 158 on the integer line with only cell 0 on. At time n form the decimal integer a(n) whose digits are the successive states of cell 0 from time 0 through n, writing 1 for on and 0 for off. For $n \geq 5$, $a(n)=10a(n-1)+a(n-4)-10a(n-5)$. The ordinary generating function, the series with coefficient a(n) on x^n, is $(1+x+x^2)/[(1-x)(1+x)(1-10x)(1+x^2)]$. [1]

Proof

A cell reads its left neighbor, itself and its right neighbor in the preceding row. Rule 158 gives output 1 on the five triples 001, 010, 011, 100 and 111, and gives 0 on the other three. Cells outside the positions from $-t$ through t stay off at time t.

Write successive row strings from left to right, with an exponent on a block denoting repeats. At time 2k the row is $(1110)^k1$; at time 2k+1 it is $111(0011)^k$. Here is a direct induction check. Pad a row at both ends by two zeros and apply the local rule to each overlapping triple. The initial row 1 gives 111, then 11101. In an even row, inserting one more block 1110 just before its final 1 replaces the old tail 1 by 11101. The output gains the four bits 0011 at the right. In an odd row, appending 0011 adds the four bits 1101 after the old output's last bit; equivalently this replaces its final 1 by 11101. These tail checks use only the last two old bits and the two padding zeros. Thus the two row formulas hold for all k.

At position zero the bit is 0 just when t has remainder 3 on division by 4. Indeed, its index counting from the left is t, starting at index zero. Let d(n) denote this bit. Appending a decimal digit gives $a(n)=10a(n-1)+d(n)$ for $n \geq 1$. Since $d(n)=d(n-4)$, subtract the same equation four times earlier to get the claimed recurrence.

The digit series is $(1+x+x^2)/(1-x^4)$, by adding the successive four-term blocks. The number a(n) weights past digits by 1, 10, 100 and so on, so its series is the digit series times $1/(1-10x)$. Factoring $1-x^4=(1-x)(1+x)(1+x^2)$ gives the formula. [4]

Context and attribution

Robert Price contributed OEIS A265380. Michael De Vlieger recorded a period-four program in 2015. Colin Barker proposed the recurrence and generating function in 2015 and 2019. The project's Lean statement is due to snoozingnewt. [1, 2]

References

[1] R. Price, M. De Vlieger and C. Barker, OEIS A265380
[2] snoozingnewt, Lean statement of A265380, submission 963
[3] Left to Solve · statement of the submitted problem
[4] Open Math Agents · merged proof, pull request #6340