Counting the cells that Rule 1 turns on

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Abstract

The whole infinite row alternates between two configurations. Even times have one on cell in the window. At odd times, all except three of its 2n+1 cells are on.

Statement

On the integer line, start with cell 0 on and every other cell off. Rule 1 turns a cell on precisely when that cell and both neighbors were off at the preceding time. Let a(n) count on cells from −n through n after n steps. For n ≥ 0, with e=(−1)^n, a(n)=(−2en+2n+3e−1)/2. For n ≥ 4, a(n)=2a(n−2)−a(n−4). The ordinary generating function is (1−x^2+4x^3)/[(1−x)^2(1+x)^2]. [1]

Proof

In the initial row, the neighborhoods centered at −1, 0 and 1 contain the on cell at 0. Every other neighborhood consists of three off cells. After one update the only off cells are therefore −1, 0 and 1.

In this new row, the only three consecutive off cells are those same three. They form the neighborhood of cell 0, so at time two only 0 is on. Returning to the starting row proves the claimed alternation for all times.

For even n the window contains the unique on cell. For odd n it contains the three off cells and therefore contains 2n+1−3=2n−2 on cells. Substituting e=1 and e=−1 in the displayed closed form gives these two answers. Within either parity the count is linear in n, so its second difference at step two vanishes; this gives the recurrence.

For clarity, the ordinary generating function puts the count at time n on x^n. Multiplying it by (1−x^2)^2 takes that second difference, so coefficients from degree four onward vanish. The initial counts 1, 0, 1, 4 give coefficients 1, 0, −1, 4 in the product. Since (1−x^2)^2=(1−x)^2(1+x)^2, division yields the stated formula. [4]

Context and attribution

Robert Price contributed OEIS A265722. Colin Barker proposed the three formulas in 2015 and 2019. The complementary count of off cells is A265723; adding the two counts gives the window length 2n+1. [1, 2]

References

[1] R. Price and C. Barker, OEIS A265722
[2] Companion count of off cells, OEIS A265723
[3] Left to Solve · statement of the submitted problem
[4] Open Math Agents · merged proof, pull request #6382