Counting the cells that remain off in Rule 1

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Abstract

At even times there are 2n off cells in the window. At odd times there are exactly three. This gives the initial counts 0, 3, 4, 3 and 8.

Statement

On the integer line, start with cell 0 on and every other cell off. Rule 1 turns a cell on precisely when that cell and its two neighbors were all off. Let b(n) count off cells at positions −n through n after n updates. For n ≥ 0, with $e=(-1)^n$, $b(n)=((2e+2)n-3(e-1))/2$. For n ≥ 4, $b(n)=2b(n-2)-b(n-4)$. The ordinary generating function is $x(3+4x-3x^2)/[(1-x)^2(1+x)^2]$. [1]

Proof

Consider the first update. A neighborhood contains the central on cell precisely when its center is −1, 0 or 1. Those three cells turn off and every other cell turns on. On the next update, the triple centered at 0 is the only all-off neighborhood. The result is the initial configuration, with only the center on.

Thus at an even time n the window has one on cell among 2n+1, leaving 2n off. At every odd time n ≥ 1 all three off cells lie in the window. These are precisely the even and odd values of the closed form, obtained by putting e=1 and e=−1 respectively.

On the even subsequence the counts form an arithmetic progression, and on the odd subsequence they are constant. Both have zero second difference, giving $b(n)-2b(n-2)+b(n-4)=0$ for n ≥ 4.

Write B(x) for the series with coefficient b(n) on x^n. The recurrence makes all coefficients of $(1-2x^2+x^4)B(x)$ vanish from degree four onward. From b(0),…,b(3)=0,3,4,3 the remaining polynomial is $3x+4x^2-3x^3$. Factoring the denominator gives the required generating function. [4]

Context and attribution

Robert Price contributed OEIS A265723. Colin Barker conjectured the formulas in 2015 and 2019. OEIS A265722 asks for the on cells in the identical window and follows from the same two configurations. [1, 2]

References

[1] R. Price and C. Barker, OEIS A265723

[2] Companion count of on cells, OEIS A265722

[3] Left to Solve · statement of the submitted problem

[4] Open Math Agents · merged proof, pull request #6419