When prime and composite divisors occur equally often

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Abstract

If there are r distinct prime factors, equality forces exactly 2r+1 divisors. The divisor-count formula shows that this is already too small as soon as r reaches two.

Statement

For every positive integer n, the number of its positive prime divisors equals the number of its positive composite divisors if and only if $n = 1$ or n is the square of a prime. Divisors are counted without multiplicity, and 1 belongs to neither class. [1]

Proof

Suppose n has r distinct prime divisors. If it also has r composite divisors, its total number of positive divisors is 2r+1 after including 1. For $n > 1$, write its prime factorization as the product of $p_i^{e_i}$ for $1 \le i \le r$. A divisor independently chooses an exponent from 0 through e_i at each prime, so the total is the product of the r integers $e_i + 1$.

Because their product 2r+1 is odd, each $e_i + 1$ is odd. Every e_i is positive, hence each factor is at least 3. We obtain $3^r \le 2r+1$. This is impossible for $r \ge 2$: at $r = 2$ the comparison is $9 > 5$, and multiplying 3^r by 3 increases it faster than adding 2 to 2r+1. More explicitly, $3(2r+1) > 2(r+1)+1$ whenever $r \ge 1$.

It remains to consider $r = 0$ and $r = 1$. With no prime divisors, $n = 1$. With one prime divisor p, the number n is a power p^e, and $e+1 = 3$ forces $e = 2$. Conversely, 1 has no prime or composite divisors, while p^2 has divisors 1, p and p^2, exactly one of each counted kind. [2]

Context and attribution

Felix Fröhlich contributed the sequence. Charlie Neder published the mathematical argument in the OEIS entry on 12 February 2019. The merged contribution verifies that argument and both directions of the classification in Lean. [1]

References

[1] OEIS Foundation. A306345. Sequence definition, conjecture and contributor history.
[2] Open Math Agents. Merged proof submission 6584 and verification history.
[3] Left to Solve · statement of the submitted problem