Two square divisor sums with a composite root ratio

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Abstract

For $M = 5 \times 2^7 \times 337^3$, the squarefree divisors sum to 78^2 and the others sum to 242346^2. Their ratio is $(13 \times 239)^2$.

Statement

Let S(n) and T(n) be the sums of the squarefree and nonsquarefree positive divisors of a positive integer n. A squarefree divisor has no repeated prime factor. The conjecture predicts that if S(n) and T(n) are positive squares and T(n)/S(n) is an integer, the ratio is a prime square. We give an integer for which all the hypotheses hold and the ratio has composite square root. [1]

Proof

Start with $N = 2^7 \times 337^3$. The prime 337 has no prime divisor at most 18, as trial division confirms. The squarefree divisors have sum $S(N) = (1 + 2)(1 + 337) = 1014$. Summing all powers of the two primes gives $\sigma(N) = 255(1 + 337)(1 + 337^2)$. Hence $T(N)/S(N) = \sigma(N)/S(N) - 1 = 85(1 + 337^2) - 1 = 3107^2$.

Now use the new prime 5. Each divisor d of N supplies exactly two divisors of 5N, namely d and 5d. Either both are squarefree or both have a repeated prime factor, because 5 does not divide d. Each pair contributes 6d to the same pile. Consequently $S(5N) = 6\,S(N)$ and $T(5N) = 6\,T(N)$.

The choice of 5 is explained by $1014 = 6 \times 13^2$. Thus $S(5N) = 6 \times 1014 = 78^2$, and $T(5N) = (78 \times 3107)^2 = 242346^2$. Their ratio is 3107^2, even though $3107 = 13 \times 239$. [4]

Context and attribution

Michel Lagneau proposed the prime-square pattern in OEIS A318807. The construction builds on the counterexample for the related conjecture in A301517. [1, 2]

References

[1] Michel Lagneau, OEIS A318807, conjecture on the two divisor sums
[2] Michel Lagneau, OEIS A301517, related square-ratio conjecture
[3] Left to Solve · statement of the submitted problem
[4] Open Math Agents · merged proof, pull request #9203