A denominator identity for the arithmetic derivative

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Abstract

Clearing denominators gives a greatest common divisor. Examining one prime power at a time shows that the same greatest common divisor is obtained from the arithmetic derivative.

Statement

For every integer $n \geq 1$, let d be the positive denominator in lowest terms of the sum of $1/p$ over the distinct prime divisors p of n, minus $1/n$. Write $D(n)$ for the arithmetic derivative: $D(1) = 0$ and $D(n)$ is the sum of $e \times (n/p)$ over the distinct primes p dividing n, where e is the exponent of p in n. Then $n/d = \gcd(n, 1 + (n - 1)D(n))$, where gcd denotes greatest common divisor. [1]

Proof

Set S equal to the sum of n/p over the distinct primes dividing n. The fraction in the statement is $(S - 1)/n$, so the cancellation factor is $\gcd(n, S - 1)$, with the sign immaterial. Also $1 + (n - 1)D(n)$ is congruent to $1 - D(n)$ modulo n. It therefore suffices to compare $\gcd(n, S - 1)$ and $\gcd(n, D(n) - 1)$.

Fix a prime p dividing n and write e for its exponent. If $e \geq 2$, every summand n/q in S is divisible by p, even the summand with $q = p$. The same holds for $D(n)$. Thus neither $S - 1$ nor $D(n) - 1$ is divisible by p, so this prime contributes to neither gcd.

If $e = 1$, the term n/p has coefficient one in both S and $D(n)$. Every term n/q with q different from p is divisible by p. Hence S and $D(n)$ agree modulo p. Their two gcds either both contain p or both omit it. Because p occurs only once in n, higher powers cannot enter either gcd. All prime exponents in the two gcds agree. For $n = 1$ the fraction is -1 and both sides equal 1. [4]

Context and attribution

Jonathan Sondow contributed OEIS A326691. Antti Karttunen conjectured this formula following an empirical discovery by a LODA miner, as recorded in the OEIS entry. [1, 2]

References

[1] Antti Karttunen, OEIS A326691, denominator sequence and conjecture
[2] OEIS A003415, arithmetic derivative
[3] Left to Solve · statement of the submitted problem
[4] Open Math Agents · merged proof, pull request #7763