A composite square root for two divisor-sum ratios

Anmol Sahu 28 November 2026

For a positive integer n, let $S(n)$ be the sum of its squarefree divisors and let $T(n)$ be the sum of its other divisors. Michel Lagneau asked whether a positive square ratio $T(n)/S(n)$ must be the square of a prime. A related conjecture asks the same question when both sums are themselves squares. The composite root $3107 = 13 \cdot 239$ gives a counterexample to each conjecture.

The first counterexample. Take $N = 2^7 \cdot 337^3$, where 337 is prime. The squarefree divisors are $1, 2, 337, 674$, so $S(N) = 3 \cdot 338 = 1014$. Let $\sigma(N) = S(N) + T(N)$ be the sum of all divisors. Summing over the two independent prime exponents gives

$$\frac{\sigma(N)}{S(N)} = \frac{1 + 2 + \cdots + 2^7}{1 + 2} \, \frac{1 + 337 + 337^2 + 337^3}{1 + 337} = 85(1 + 337^2).$$

Subtracting one yields $T(N)/S(N) = 9{,}653{,}449 = 3107^2$. This settles the question attached to OEIS A301517.

Making both sums squares. Since $5 \nmid N$, every divisor of $5N$ has the form d or $5d$ for exactly one $d \mid N$. The two are squarefree together. Both divisor sums are therefore multiplied by six, preserving their ratio, while

$$S(5N) = 6 \cdot 1014 = 78^2, \qquad T(5N) = 78^2 \cdot 3107^2 = 242346^2.$$

Thus $5N$ supplies the counterexample under the extra hypothesis in OEIS A318807.

Credit and verification. The conjectures are due to Michel Lagneau. The counterexamples and Lean disproofs were submitted under Anmol Sahu's GitHub account with OpenAI Codex, which also drafted this note. The two formal proofs merged separately in Open Math Agents. Their public records are PR 7662 and PR 9203.