Adding one element to an additive basis

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Abstract

There are at most two new ways. Once the new number is used, the desired total determines its partner.

Statement

For a set A of nonnegative integers and an integer n ≥ 0, let r(A,n) count the ordered pairs (x,y) from A with x+y=n. For every integer a ≥ 0, adjoining a to A gives r(A ∪ {a},n) ≤ r(A,n)+2. [1]

Proof

Fix n and examine a pair that becomes available after adjoining a. If neither coordinate equals a, both coordinates already belong to A, so the pair was available before. Every new pair must therefore use a.

If a exceeds n, no pair of nonnegative summands of n can use a. Otherwise the only possible partner is n−a. The candidates are (a,n−a) and (n−a,a). They coincide when n=2a, and they contribute nothing if the partner is unavailable. Thus at most two pairs are new.

The constant can be attained. With A={2}, a=3 and n=5 the count rises from zero to two, because 2+3 and 3+2 are both available. Applying the same bound repeatedly shows that adjoining k elements increases each count by at most 2k. [5]

Context and attribution

The Erdős–Turán conjecture asks whether, whenever every sufficiently large integer is a sum of two elements of A, the numbers of such representations are unbounded. The estimate here is a lemma in TheWaler's reduction. An earlier submission also addressed this lemma. [1, 2, 3]

References

[1] Erdős Problems, problem 28
[2] TheWaler, reduction of problem 28, submission 4711
[3] Earlier submission on the same lemma, 4900
[4] Left to Solve · statement of the submitted problem
[5] Open Math Agents · merged proof, pull request #5250