Why covering through 70 forces six ordered representations

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Abstract

The last possible endpoint is 69. An explicit set works there. A finite search certified in Lean eliminates every choice at 70.

Statement

Let S be any subset of {0,1,...,70}. Assume that for every integer t from 0 through 70 some a,b in S satisfy a+b=t. Then some t in the same interval has at least six distinct ordered pairs (a,b) from S with that sum. At endpoint 69 a set exists for which every count lies between one and five. [2]

Proof

For $t \geq 0$ write c(t) for the number of ordered pairs of chosen elements with sum t. To search up to an endpoint L, decide membership in order from 0 to L. Zero is forced because the only way to make 0 is 0+0.

At the stage for x, try both choices, leaving x out and putting x in. If x is put in, every previously chosen b contributes two new pairs of sum x+b, and (x,x) contributes one of sum 2x. Ignore sums above L. Discard the branch if any count exceeds five. Also discard it if c(x)=0 after the decision: every undecided integer exceeds x and cannot help to represent x. Retain all other branches.

These rules never discard a set satisfying both requirements. Conversely, a branch that reaches L has tested coverage at every total and kept every count at most five. Checking the two continuations at each surviving branch therefore decides the question exactly. For L=70 the certified search has no surviving branch. Its Lean implementation verifies 42 subtrees and checks six distinct first summands whenever it discards a branch for exceeding the cap. The second summand is determined by the total.

For L=69 use S={0,1,3,4,5,6,10,15,17,23,31,39,47,55,63}. Counting the pairs directly gives respectively 4, 42, 3, 17 and 4 totals with one, two, three, four and five representations. These frequencies add to 70, and the four totals attaining five are 6, 10, 20 and 62. This proves sharpness. [5]

Context and attribution

Grekos, Haddad, Helou and Pihko published the endpoint in 2003, §6.4. TheWaler independently recovered it on the worksheet. The submitted contribution is a machine-checked proof and a concrete set showing sharpness. [1, 2, 3]

References

[1] G. Grekos, L. Haddad, C. Helou and J. Pihko, On the Erdős–Turán conjecture, J. Number Theory 102 (2003), §6.4

[2] Erdős Problems, problem 28

[3] Reduction to finite bounds, submission 5979

[4] Left to Solve · statement of the submitted problem

[5] Open Math Agents · merged proof, pull request #7608