A Pell-equation proof for quotients of consecutive products

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Abstract

Yes. Rewriting each consecutive product as a difference of squares turns the question into a Pell equation. Its positive solutions give pairs arbitrarily far apart.

Statement

For every nonsquare integer $N \geq 2$ there are integers $k \geq 2$, $n \geq 0$ and $m \geq n+k$ such that N equals the product of $m+1$ through $m+k$ divided by the product of $n+1$ through $n+k$. In fact $k=2$ suffices. [2]

Proof

Write $X=2m+3$ and $Y=2n+3$. The identity $4(m+1)(m+2)=X^2-1$ converts the desired equality into $X^2-NY^2=1-N$. It is enough to construct odd positive solutions with $Y \geq 3$ and $X-Y \geq 4$.

Pell's theorem for positive nonsquare N gives positive integers u,v with $u^2-Nv^2=1$. Squaring $u+v\sqrt{N}$ gives $U+V\sqrt{N}$, where $U=u^2+Nv^2=1+2Nv^2$ is odd and $V=2uv$ is even. This square also has norm one, meaning $U^2-NV^2=1$.

Start at $X=Y=1$. Repeatedly replace (X,Y) by $(UX+NVY,VX+UY)$. The expression X^2-NY^2 is multiplied by $U^2-NV^2=1$, and both coordinates stay positive and odd. Since $U>1$, the successive values of Y grow without bound.

From $X^2=NY^2+1-N$ we have X/Y tending to $\sqrt{N}$ as Y grows. Because $\sqrt{N}>1$, eventually $X-Y \geq 4$ and $Y \geq 3$. Setting $m=(X-3)/2$ and $n=(Y-3)/2$ now gives nonnegative integers with $m \geq n+2$, and the displayed identity gives the desired ratio. [5]

Context and attribution

Erdős and Straus gave the Pell-equation argument in 1977. The project submission adapts the existing Formal Conjectures proof and changes its two small numerical checks to kernel-verified computations. The square case is a separate part of problem 686. [1, 2, 3]

References

[1] P. Erdős and E. G. Straus, On Products of Consecutive Integers (1977), p. 63
[2] Erdős Problems, problem 686
[3] Formal Conjectures, formalization of Erdős problem 686
[4] Left to Solve · statement of the submitted problem
[5] Open Math Agents · merged proof, pull request #5821