Three fixed red sites in a 158-site coloring

Anmol Sahu · 28 November 2026

Abstract

To avoid a red triple, two red sites can force a third site blue. To avoid a blue progression of length thirteen, a nearly blue progression can force its last site red. A finite certificate checks that these rules and exhaustive case splits rule out every completion.

Statement

Color each integer from 0 through 157 either red or blue, with 76, 78 and 79 colored red. Then there is either a red arithmetic progression of length three or a blue arithmetic progression of length thirteen. An arithmetic progression has the form a, a + d, a + 2d, and so on, where d is a positive integer. [1]

Proof

Assume a coloring avoids both forbidden patterns. Keep track of sites already chosen red and sites still allowed to be red. Two known red sites prohibit any third site that completes a red triple. For example, 76 and 78 prohibit 74 and 80, as well as their midpoint 77. Each of these deductions follows from an actual progression on the board.

Every thirteen-term progression must still have a red site. Its possible spacings d range from 1 through 13, since twelve gaps must fit within 157. For each spacing, all starts from 0 through 157 − 12d are possible. If a progression has no allowed red site, the current case closes. If it has exactly one and no known red site, that site must be red. With several candidates left, choose one and examine both possibilities, red and blue.

These rules give a finite proof by cases. The certificate supplies the choices and previously established subcases. Its Lean verifier proves that a successful traversal excludes every coloring consistent with the starting state. In the deposited source the theorem visit_sound proves the traversal rule, the lemmas c774 through c800 check the 27 certificate pieces, and solvedB joins them at the starting state with 76, 78 and 79 red. The final theorem closeSecondWindow transfers this conclusion to the stated coloring problem.

The proof of the finite traversal is reproducible from the signed Lean file cited below, which contains the complete certificate data. The contradiction excludes a coloring that avoids both progressions. [7]

Context and attribution

The site's Green 14 problem concerns van der Waerden numbers. The formal argument leading to this case uses reductions by NickLeko, ZedingZhang and anmolspace. The companion branch specifies two blue sites. The submitted certificate for the three-red branch adapts a propagation argument from submission 4877 and credits earlier work by MrSeizy, Vincent T. Mossman and PanPapag. [1, 2, 3, 4, 5]

References

[1] Ben Green, 100 open problems, problem 14
[2] B. Landman, A. Robertson and C. Culver, Some New Exact van der Waerden Numbers (2005)
[3] Reduction to the red and blue branches, PR 8422
[4] Anmol Sahu, complete three-red certificate and verifier
[5] Earlier coloring-propagation argument, PR 4877
[6] Left to Solve · statement of the submitted problem
[7] Open Math Agents · merged proof, pull request #9049